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ASYMPTOTIC FINITE ELEMENT METHOD FOR BOUNDARY VALUE PROBLEMS

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SUMMARY

The Asymptotic Finite Element method for improvement of standard finite element solutions of perturbation equations by the addition of asymptotic corrections to the right hand side terms is presented. It is applied here to 1-D and 2-D diffusion–convection equations and to non-linear similarity equations. Excellent results were obtained without the *a priori* use of special trial and test functions. Theoretical expectations were confirmed.

1. INTRODUCTION

Regular and singular perturbation problems arise in different fields of interest including solid and fluid mechanics, heat transfer, etc.[1–4] Approximate analytical solutions are available through asymptotic expansions techniques which consist of straightforward series expansions in the perturbation parameter for regular perturbation problems and a 'composite' construction of 'inner' and 'outer' solutions in the case of singular perturbation problems. The accuracy of these approximations is of $O(\varepsilon^{n+1})$ (where ε is the small characteristic parameter and n is the order of the asymptotic approximation). For boundary layer problems, these asymptotic approximations can recover the solution behaviour within thin layers. In practice only the first few terms in the asymptotic approximation are calculated because of complexity and the fact that the series is asymptotic and not convergent in general. This limits the usefulness of asymptotic techniques to values of ε which may not be of interest over the range required in the application. For moderate values of the perturbation parameter, the accuracy of asymptotic approximations is usually insufficient. However, for these cases standard finite element schemes can give good results even when a crude mesh of elements is employed. For stiff problems (with thin boundary layers), standard numerical schemes can not exhibit solutions which are a good approximation to the mathematical model. To overcome this difficulty, several approaches are possible, such as the use of non-uniform meshes, adaptive techniques, upwind schemes, etc.[5–7] The applicability and accuracy of such schemes are presently open questions.

The asymptotic information on the solution behaviour has been used for constructing finite element schemes of different types. One approach is motivated by the classical singular perturbation methods where inner and outer *numerical* solutions are combined to give approximate solutions. These solutions become more accurate as the equations become stiffer; however, the error estimate of these schemes includes a term which is fixed for a given ε and for a

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given problem one has no numerical control on the desired level of accuracy for the model problem. A different approach is based on constructing trial and test space of functions which incorporate exponential basis functions that can asymptotically recover the solution behaviour at the boundary layers.

In the present work we introduce the so-called Asymptotic Finite Element method which circumvents the difficulty of *a priori* constructing special trial and test functions, and yet can *a posteriori* recover the asymptotic structure of the solution. By using this approach one can improve the accuracy of both asymptotic and standard numerical solutions.[8,9] It is interesting to point out that the correction and interpolation steps of the present method are related to similar steps of certain multi-grid versions and can be viewed as a continuous analogue of them.[10]

The main aims of the present work are to get *a posteriori* verification to the predicted global error estimates,[9] and to present an extension of the proposed scheme to multi-dimensional diffusion–convection equations.

2. FORMULATION

2.1 Statement of the problem

Let Ω be a bounded open set of R^N with boundary Γ. We consider the following boundary value problem:

$$Lu = f, \quad \text{in } \Omega, \tag{1}$$

$$u = g, \quad \text{on } \Gamma, \tag{2}$$

where L denotes a given elliptic differential operator. It is assumed that L is a well behaved operator and f and g are assumed to be sufficiently smooth functions of $\mathbf{x} \in \bar{\Omega} = \Omega + \Gamma$. In the above equations, L, f and g can be a function of a small positive characteristic parameter denoted by ε.

2.2 Asymptotic approximation

Suppose that we have at our disposal an asymptotic solution of equations (1) and (2) of the following form:

$$u^\varepsilon \sim \sum_{i=0}^{n} u_i(\mathbf{x}; \varepsilon)\varepsilon^i, \tag{3}$$

where u_i can be composed of inner and outer parts in the case of singular perturbation problems (more complicated constructions can also be accommodated). This solution is called the nth order asymptotic approximation, since $e^\varepsilon = u - u^\varepsilon = O(\varepsilon^{n+1})$. Let f^ε be obtained by operating with L on u^ε and g^ε be an asymptotic approximation to the boundary conditions, i.e.

$$f^\varepsilon = Lu^\varepsilon, \quad \text{in } \Omega, \tag{4}$$

$$g^\varepsilon = u^\varepsilon, \quad \text{on } \Gamma. \tag{5}$$

2.3 Finite element formulations

Let the given domain $\bar{\Omega}$ be approximated by $\bar{\Omega}^h$ which consists of an assemblage of finite elements.[11] The representation of u within $\bar{\Omega}^h$ is approximated by

$$u \sim u^h = N_i(\mathbf{x})u_i^h = \mathbf{N}\mathbf{U}^h, \quad \mathbf{x} \in \bar{\Omega}^h, \tag{6}$$

where $u_i^h = u^h(\mathbf{x}_i)$ is the value of the approximate solution at the ith nodal point, N_i is the corresponding global trial function, $\mathbf{N}$ is the global matrix of trial functions and $\mathbf{U}^h$ is a vector whose entities are u_i^h (here and in the sequel, we use the summation convention with summation over the nodes within the given region). The strong form, Petrov–Galerkin finite element formulation of (1) can be written as the following matrix equation:

$$(\mathbf{M}^{\mathrm{T}}, L\mathbf{N})_h \mathbf{U}^h = (\mathbf{M}^{\mathrm{T}}, f)_h, \tag{7}$$

where $\mathbf{M}$ is the global matrix of test functions, and $(\cdot,\cdot)$ denotes the usual inner product in $L_2(\Omega)$. The index h in $(\cdot,\cdot)_h$ denotes an approximation to $(\cdot,\cdot)$ obtained by a quadrature rule. In this formulation the trial space of functions is too restricted. To relax the smoothness requirement on the trial functions one applies the Green's theorem in such an optimal way that both trial and test functions have the same smoothness requirements. This yields the so-called weak formulation which, after imposing the boundary conditions (2), gives the following matrix equation:

$$\mathbf{A}^h \mathbf{U}^h = \mathbf{f}^h. \tag{8}$$

Our Asymptotic Finite Element (AFE) scheme of (1) is defined by

$$\mathbf{A}^h(\bar{\mathbf{U}}^h)\bar{\mathbf{U}}^h = \mathbf{f}^h + [\mathbf{A}^h(\mathbf{U}^\varepsilon)\mathbf{U}^\varepsilon - \mathbf{f}^{\varepsilon,h}], \tag{9}$$

where $\mathbf{U}^\varepsilon$ is a vector whose entities are $u_i^\varepsilon = u^\varepsilon(\mathbf{x}_i)$ and $\mathbf{f}^{\varepsilon,h} = (\mathbf{M}^{\mathrm{T}}, f^\varepsilon)_h$. Here, the terms outside the square brackets coincide with the Standard Finite Element (SFE) scheme (8), while the terms inside the square brackets represent the correction terms which are the essence of the present AFE scheme. For a linear operator L, equation (9) is equivalent to

$$\mathbf{A}^h \bar{\mathbf{e}}^\varepsilon = \mathbf{f}^h - \mathbf{f}^{\varepsilon,h}, \tag{10}$$

where

$$\bar{\mathbf{e}}^\varepsilon \equiv \bar{\mathbf{U}}^h - \mathbf{U}^\varepsilon. \tag{11}$$

Hence, for this case, the AFE scheme is equivalent to the application of the standard parent method to the asymptotic error $f - f^\varepsilon$. This gives rise to a correction term which is then added to u^ε. This correction term should vanish in the limit of $\varepsilon \to 0$ or $h \to 0$. Applying the standard interpolation (6), using $\bar{\mathbf{U}}^h$ instead of $\mathbf{U}^h$ will not give improved results within the elements, since a polynomial interpolation can not recover the solution behaviour, especially when there are no nodal points inside the boundary layers. Instead, we apply the following *a posteriori* global approximation:

$$\tilde{u} = u^\varepsilon + \mathbf{N}\bar{\mathbf{e}}^\varepsilon, \tag{12}$$

which can recover the solution behaviour. Based on the above equations, we can establish,[9] for appropriate linear operators L and under the assumption that the exact and asymptotic solutions are sufficiently smooth, the following theorem.

Theorem. Let us have an SFE error estimate of the form

$$\| e^{\mathrm{SFE}} \| = \| u - u^h \| < C_1 F(h, \varepsilon). \tag{13}$$

Then, for the AFE scheme, we have the following error estimate:

$$\| e^{\mathrm{AFE}} \| = \| u - \bar{u}^h \| < C_2 \varepsilon^{n+1} F(h, \varepsilon), \tag{14}$$

where $\|\cdot\|$ is either the Euclidean norm or the maximum norm denoted by $\|\cdot\|_0$ and $\|\cdot\|_\infty$, respectively and C_i denotes a generic constant independent of h and ε. Usually, the numerical error estimate is dependent on the norm of the $k+1$ first derivatives of the exact solution, where k is the degree of the highest complete polynomial interpolation. For several linear versions of (1), the error

estimate of finite element solutions employing linear elements can be bounded by either the Euclidean or maximum norm of the right hand side. For these cases, we assume that for small values of h and ε, the following estimates hold:

$$\| e^{\mathrm{SFE}} \|_i \sim C_3 H(h) \| f \|_i, \tag{15}$$

$$\| e^{\varepsilon} \|_i \sim C_4 E(\varepsilon) \| f \|_i, \quad i = 0, \infty, \tag{16}$$

$$\| e^{\mathrm{AFE}} \|_i \sim C_5 E(\varepsilon) H(h) \| f \|_i, \tag{17}$$

where H and E are given functions and C_3, C_4 and C_5 are positive constants. Then, for a given mathematical problem and numerical schemes, the ratio defined by

$$K_i = \frac{\| e^{\mathrm{AFE}} \|_i \| f \|_i}{\| e^{\mathrm{SFE}} \|_i \| e^{\varepsilon} \|_i}, \quad i = 0, \infty, \tag{18}$$

approaches a limit value. These assumptions may not hold in particular cases. However, they do seem to hold in non-trivial cases.

The AFE scheme has the following elegant features.

(a) By using standard finite element and asymptotic approximations, one gets an AFE scheme which can improve the accuracy of both the SFE scheme—by a factor of $O(\varepsilon^{n+1})$—and the asymptotic approximation—by a factor of the order of the discretization error of the parent SFE scheme.

(b) The AFE scheme is constructed from the SFE scheme through a modification of the right hand side vector. Therefore:

(i) it can be easily adjusted to any existing SFE code, since the trial and test functions are the same as those employed for the parent SFE scheme. Moreover, by using the *a posteriori* interpolation (12), one can easily recover the solution behaviour and circumvent the difficulty of *a priori* constructing trial and test spaces of functions which incorporate special functions;

(ii) applying certain SFE schemes, like the Bubnov–Galerkin scheme, to stiff problems, results in physically unrealistic oscillatory solutions unless the mesh is appropriately constructed. These wiggles appear because of the unfavourable properties of $(\mathbf{A}^h)^{-1}$, which are not improved by the AFE scheme. Therefore, it is not clear that the AFE scheme can eliminate this phenomenon, but it is expected that the amplitude of these wiggles will be reduced by a factor of $O(\varepsilon^{n+1})$. Thus, the AFE scheme can be effective for the whole range of h/ε;

(iii) the round-off error is about the same as in the parent scheme. This property is of crucial importance for penalty finite element schemes, where the round-off error is proportional to the value of the penalty parameter.[12] By using the AFE scheme, one can reduce the round-off error, since for a given accuracy, with this scheme, one can use smaller values of the penalty parameter;[13]

(iv) the AFE scheme yields a global set of algebraic equations with the same dimension as the set yielded by the parent SFE scheme. Hence, by using the AFE scheme, one can either, for given computer facilities, get a more accurate solution, or for a given accuracy, use smaller computer facilities.

3. SAMPLE PROBLEMS

Here we shall restrict the discussion to three examples, where the main features of the AFE method are already exhibited. The following problems have been tested.

3.1 *Linear 1-D stiff equations*

Let us consider the following problem:

$$Lu = \varepsilon u'' + qu' + ru = f, \quad \text{in } (0, 1), \tag{19}$$

$$u(0) = 0, \quad u(1) = 0, \tag{20}$$

where q, r and f are given functions of x which are assumed to be sufficiently smooth, and $(\ )'$ denotes differentiation with respect to the coordinate x. Suppose that the one dimensional region $[0, 1]$, is discretized by finite elements. We consider a partition Π of $(0, 1)$:

$$\Pi = \{0 = x_0 < x_1 < x_2 \cdots < x_{m+1} = 1\}, \tag{21}$$

where m is the number of inner nodal points, and define the maximum discrete error norm and the Euclidean error norm by

$$e_\infty = \|u - \hat{u}\|_{\Pi,\infty} \equiv \max_{x_i \in \Pi} |u_i - \hat{u}_i|, \tag{22}$$

$$e_0 = \|u - \hat{u}\|_0 \equiv \left[\int_\Omega (u - \hat{u})^2 d\Omega\right]^{1/2}, \tag{23}$$

where u is a reference solution which is considered to be either the exact solution when it is available or the best numerical solution available and $\hat{u}$ represents a given approximate solution.

The Petrov–Galerkin finite element formulation of (19) yields the following matrix equation:

$$\begin{aligned}&[-\varepsilon(\mathbf{M}'^{\mathrm{T}}, \mathbf{N}')_h + (\mathbf{M}^{\mathrm{T}}, q\mathbf{N}' + r\mathbf{N})_h]\mathbf{U}^h \\ &\quad = (\mathbf{M}^{\mathrm{T}}, f)_h + \lambda\{[-\varepsilon(\mathbf{M}'^{\mathrm{T}}, \mathbf{N}')_h + (\mathbf{M}^{\mathrm{T}}, q\mathbf{N}' + r\mathbf{N})_h]\mathbf{U}^\varepsilon - (\mathbf{M}^{\mathrm{T}}, f^\varepsilon)_h\},\end{aligned} \tag{24}$$

where $\lambda = 0$ for the SFE scheme and $\lambda = 1$ for the AFE scheme. The overbar which was used to distinguish the AFE solution from the SFE solution has been omitted. The context affords sufficient means to avoid confusion.

Test case. As a particular example we examine the following stiff problem:

$$\varepsilon u'' + u' + u = -(1 + x), \quad \text{in } (0, 1), \tag{25}$$

$$u(0) = 0, \quad u(1) = 0. \tag{26}$$

The first order asymptotic solution is given by

$$u^\varepsilon = e^{1-x} - e^{1+x-x/\varepsilon} - x + \varepsilon[(1 - x)e^{1-x} - (1 + x)e^{1+x-x/\varepsilon}]. \tag{27}$$

This example was run with $\varepsilon = h^\gamma$ for various values of γ, using the SFE and AFE schemes. Both Bubnov–Galerkin and Petrov–Galerkin formulations have been employed.[14] The region has been discretized by a uniform mesh of linear elements. For each value of γ, the element length, h, was successively halved. The L_2 global error norm and the appropriate rate of convergence, R_0, for the above example, are shown in Tables I–III. For $\gamma < 1$, the performance of the Bubnov–Galerkin scheme is better than that of the Petrov–Galerkin scheme, while for $\gamma > 1$ the Petrov–Galerkin scheme is superior. As expected, the performance of the AFE scheme shows that the improvement of the rate of convergence, $R_0^{\mathrm{AFE}} - R_0^{\mathrm{SFE}}$, approaches $(n + 1)\gamma$. Similar conclusions have been found regarding the results with respect to the maximum discrete norm.[9,15] In order to get *a posteriori* verification of the theorem, the K_i ratio has been computed. The values of K_0 and K_∞ are shown in Table IV. As expected, these ratios approach limit values which are of $O(1)$. The above results

Table I. Example 1a—comparison between SFE and AFE schemes (Bubnov–Galerkin, zeroth order of asymptotic approximation)

NE	γ	e_0^e	e_0^h		R_0	
			SFE	AFE	SFE	AFE
64	0·5	1·19E-1	6·84E-4	8·13E-5	1·11	1·63
	0·75	5·25E-2	3·43E-3	1·50E-4	0·851	1·58
	1·0	1·94E-2	1·62E-2	2·52E-4	0·491	1·49
	1·5	2·46E-3	1·74E-1	3·40E-4	0·116	1·62
	2·0	3·09E-4	6·35E-1	1·55E-4	−9·77E-3	1·99
128	0·5	9·45E-2	2·97E-4	2·54E-5	1·20	1·68
	0·75	3·21E-2	1·89E-3	4·94E-5	0·860	1·60
	1·0	9·79E-3	1·15E-2	8·95E-5	0·496	1·49
	1·5	8·72E-4	1·54E-1	1·07E-4	0·171	1·67
	2·0	7·71E-5	6·36E-1	3·88E-5	−2·59E-3	2·00
256	0·5	7·15E-2	1·27E-4	7·76E-6	1·23	1·71
	0·75	1·94E-2	1·04E-3	1·62E-5	0·866	1·61
	1·0	4·92E-3	8·12E-3	3·17E-5	0·498	1·50
	1·5	3·09E-4	1·34E-1	3·27E-5	0·205	1·67
	2·0	1·93E-5	6·37E-1	9·72E-6	−6·68E-4	2·00
512	0·5	5·25E-2	5·38E-5	2·35E-6	1·24	1·72
	0·75	1·16E-2	5·68E-4	5·26E-6	0·870	1·62
	1·0	2·46E-3	5·75E-3	1·12E-5	0·499	1·50
	1·5	1·09E-4	1·15E-1	9·89E-6	0·226	1·73
	2·0	4·77E-6	6·37E-1	2·43E-6	−1·70E-4	2·00

Table II. Example 1b—comparison between SFE and AFE schemes (Bubnov–Galerkin, first order of asymptotic approximation)

NE	γ	e_0^e	e_0^h		R_0	
			SFE	AFE	SFE	AFE
64	0·5	3·87E-2	6·84E-4	2·65E-5	1·11	2·39
	0·75	5·61E-3	3·43E-3	1·66E-5	0·851	2·33
	1·0	7·31E-4	1·62E-2	9·86E-6	0·491	2·49
	1·5	1·16E-5	1·74E-1	1·66E-6	0·116	3·12
	2·0	1·73E-7	6·35E-1	9·47E-8	−9·77E-3	3·99
128	0·5	2·03E-2	2·97E-4	5·59E-6	1·20	2·24
	0·75	2·04E-3	1·98E-3	3·24E-6	0·860	2·35
	1·0	1·84E-4	1·15E-2	1·75E-6	0·496	2·50
	1·5	1·38E-6	1·54E-1	1·84E-7	0·171	3·17
	2·0	1·08E-8	6·36E-1	5·92E-9	−2·59E-3	4·00
256	0·5	1·08E-2	1·27E-4	1·21E-6	1·23	2·21
	0·75	7·31E-4	1·04E-3	6·31E-7	0·866	2·36
	1·0	4·63E-5	8·12E-3	3·10E-7	0·498	2·50
	1·5	1·73E-7	1·34E-1	2·00E-8	0·205	3·21
	2·0	6·74E-10	6·37E-1	3·70E-10	−6·68E-4	4·00
512	0·5	5·61E-3	5·38E-5	2·59E-7	1·24	2·22
	0·75	2·60E-4	5·68E-4	1·22E-7	0·870	2·37
	1·0	1·16E-5	5·75E-3	5·48E-8	0·499	2·50
	1·5	2·16E-8	1·15E-1	2·14E-9	0·226	3·23
	2·0	4·21E-11	6·37E-1	2·32E-11	−1·70E-4	4·00

Table III. Example 1c—comparison between SFE and AFE schemes (Petrov–Galerkin, zeroth order of asymptotic approximation)

NE	γ	e_0^ε	e_0^h		R_0	
			SFE	AFE	SFE	AFE
	0·5	1·19E-1	2·98E-3	3·59E-4	1·18	1·72
	0·75	5·25E-2	1·25E-2	5·54E-4	0·905	1·64
64	1·0	1·94E-2	4·63E-2	7·26E-4	0·523	1·52
	1·5	2·46E-3	1·55E-1	3·04E-4	0·366	1·87
	2·0	3·09E-4	1·90E-1	4·65E-5	0·457	2·46
	0·5	9·45E-2	1·22E-3	1·05E-4	1·29	1·77
	0·75	3·21E-2	6·78E-3	1·78E-4	0·888	1·63
128	1·0	9·79E-3	3·25E-2	2·54E-4	0·512	1·51
	1·5	8·72E-4	1·17E-1	8·12E-5	0·402	1·90
	2·0	7·71E-5	1·37E-1	8·34E-6	0·479	2·48
	0·5	7·15E-2	4·95E-4	3·07E-5	1·30	1·78
	0·75	1·94E-2	3·68E-3	5·76E-5	0·880	1·63
256	1·0	4·92E-3	2·29E-2	8·95E-5	0·506	1·51
	1·5	3·09E-4	8·72E-2	2·13E-5	0·430	1·93
	2·0	1·93E-5	9·74E-2	1·49E-6	0·489	2·49
	0·5	5·25E-2	2·04E-4	8·99E-6	1·28	1·77
	0·75	1·16E-2	2·01E-3	1·87E-5	0·877	1·63
512	1·0	2·46E-3	1·62E-2	3·16E-5	0·503	1·50
	1·5	1·09E-4	6·38E-2	5·51E-6	0·450	1·95
	2·0	4·77E-6	6·91E-2	2·64E-7	0·495	2·50

Table IV. Example 1—computed values of K_0 and K_∞

NE	γ	Ex. 1a		Ex. 1b		Ex. 1c	
		K_∞	K_0	K_∞	K_0	K_∞	K_0
	0·5	1·33	1·52	1·34	1·53	1·30	1·54
	0·75	0·938	1·27	0·949	1·31	0·934	1·28
64	1·0	0·821	1·23	0·828	1·27	0·821	1·23
	1·5	0·758	1·21	0·760	1·26	0·756	1·21
	2·0	0·579	1·21	0·761	1·32	0·760	1·21
	0·5	1·13	1·38	1·15	1·42	1·12	1·54
	0·75	0·865	1·24	0·874	1·29	0·866	1·25
128	1·0	0·784	1·22	0·787	1·26	0·784	1·22
	1·5	0·747	1·21	0·748	1·32	0·746	1·21
	2·0	0·747	1·21	0·748	1·32	0·748	1·21
	0·5	1·01	1·31	1·027	1·35	1·01	1·32
	0·75	0·82	1·23	0·826	1·27	0·820	1·23
256	1·0	0·763	1·21	0·765	1·26	0·763	1·21
	1·5	0·741	1·21	0·742	1·32	0·741	1·21
	2·0	0·742	1·21	0·742	1·32	0·742	1·21
	0·5	0·938	1·27	0·949	1·31	0·934	1·28
	0·75	0·792	1·22	0·795	1·26	0·791	1·22
512	1·0	0·751	1·21	0·753	1·26	0·751	1·21
	1·5	0·739	1·21	0·739	1·32	0·738	1·21
	2·0	0·739	1·22	0·739	1·32	0·739	1·22

Table V. Example 1a—solution accuracy as a function of x and NE ($\varepsilon = 10^{-3}$)

		8 element		16 element		32 element	
x	e^{ε}	e^{h}_{SFE}	$(\hat{K}-1)\times 10^{3}$	e^{h}_{SFE}	$(\hat{K}-1)\times 10^{3}$	e^{h}_{SFE}	$(\hat{K}-1)\times 10^{3}$
1/32	2·56E-3					−2·76	1·28
1/16	2·40E-3			1·68E1	1·19	2·12	1·44
3/32	2·25E-3					−2·20	1·27
1/8	2·10E-3	5·06	0·726	2·14	1·46	1·65	1·46
5/32	1·97E-3					−1·77	1·26
3/16	1·84E-3			1·51E1	1·18	1·28	1·47
7/32	1·71E-3					−1·42	1·25
1/4	1·59E-3	1·78	1·34	1·65	1·46	−9·89E-1	1·48
9/32	1·48E-3					−1·14	1·24
5/16	1·37E-3			1·35E1	1·17	7·61E-1	1·50
11/32	1·27E-3					−9·23E-1	1·24
3/8	1·17E-3	4·28	0·540	1·24	1·45	5·81E-1	1·52
11/32	1·08E-3					−7·49E-1	1·23
7/16	9·89E-4			1·21E1	1·16	4·40E-1	1·54
15/32	9·06E-4					−6·10E-1	1·23
1/2	8·26E-4	1·04	1·22	8·89E-1	1·45	3·30E-1	1·56

suggest the following strategy to estimate various errors in our computations, in particular the AFE error:

(i) calculate K_i for a relatively large value of the characteristic parameter;
(ii) calculate SFE, AFE and asymptotic solutions, for values of ε which are of interest. The error introduced in these approximations can be estimated by

$$\| e^{\varepsilon} \|_i \sim \| \tilde{u} - u^{\varepsilon} \|_i, \tag{28}$$

$$\| e^{\mathrm{SFE}} \|_i \sim \| \tilde{u} - u^{\mathrm{SFE}} \|_i, \tag{29}$$

$$\| e^{\mathrm{AFE}} \|_i \sim K_i \| e^{\varepsilon} \|_i \| e^{\mathrm{SFE}} \|_i / \| f \|_i, \quad i = 0, \infty. \tag{30}$$

It is clear that this procedure can be effective only in those cases where the AFE error is smaller than both SFE and asymptotic errors by at least one order of magnitude. The error distributions of the approximate solutions for the above model problem are shown in Table V. The pattern of the wiggles is clearly demonstrated. In order to represent the behaviour of the corresponding AFE profile, the following ratio has been defined:

$$\hat{K}_i \equiv \frac{e_i^{\mathrm{AFE}}}{e_i^{\mathrm{SFE}}{}_{\varepsilon}}, \quad i = 1, 2, \ldots, m. \tag{31}$$

The Table demonstrates that, within 0·1 per cent, the amplitude of the AFE pointwise error is smaller by a factor of ε than the corresponding SFE error. As expected, the AFE scheme does not remove the oscillations entirely, but their amplitude is considerably reduced.

3.2 *Non-linear similarity equations*

Several problems of interest can be treated by similarity methods which reduce the Navier–Stokes equations to a system of non-linear similarity equations. The AFE method can be applied to

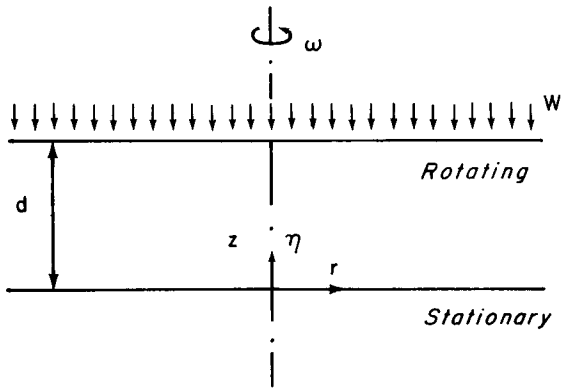

Figure 1. Flow between a stationary and a rotating disk with suction—notation

such a system of ordinary differential equations. Depending on the particular problem, regular or singular perturbation solutions can be constructed and used in the present method.[16–19] Finite element formulations of a typical form of these equations are described in detail in Reference 18 and owing to space limitations will not be presented here.

Test case—flow between a stationary and a rotating disk with suction (Figure 1). The governing differential equations for the laminar flow of a Newtonian incompressible fluid confined between two infinite disks are given by

$$F''' + Re(FF'' - 0{\cdot}5F'^2) + \alpha G^2 = A, \tag{32}$$

$$-G'' + Re(F'G - FG') = 0, \tag{33}$$

where

$$u = \frac{Wr}{2d}F'(\eta), \quad v = \omega r G(\eta),$$

$$w = -WF(\eta), \quad \eta \equiv z/d, \tag{34}$$

$$Re \equiv Wd/\nu, \qquad \alpha \equiv 2d^3\omega^2/\nu W.$$

Here, r, z, are cylindrical coordinates, u, v, w the radial, azimuthal and axial velocity components, ω is the angular velocity, Re is the suction Reynolds number, α is a rotation coefficient, and A is the constant of integration proportional to the pressure gradient, to be determined from the boundary conditions which are

$$\begin{aligned} &@\ \eta = 0: && F = F' = G = 0; \\ &@\ \eta = 1: && F = 1, \quad F' = 0, \quad G = 1. \end{aligned} \tag{35}$$

An asymptotic solution of the above equations has been developed for small values of Re. The first order solution is given by[19]

$$\begin{aligned} F^\varepsilon &= F_0 + ReF_1, \\ G^\varepsilon &= G_0 + ReG_1, \\ A^\varepsilon &= A_0 + ReA_1, \end{aligned} \tag{36}$$

where

$$F_0 = (3\eta^2 - 2\eta^3) + \frac{\alpha}{60}(-2\eta^2 + 3\eta^3 - \eta^5),$$

$$G_0 = \eta,$$

$$A_0 = -12 + 0{\cdot}3\alpha,$$

$$F_1 = (13\eta^2 - 18\eta^3 + 7\eta^6 - 2\eta^7)/70 + \alpha 10^{-5}(15\eta^2 + 8\eta^3 + 252\eta^5 - 546\eta^6 + 216\eta^7 + 45\eta^8 + 10\eta^9)/1{\cdot}512 + \alpha^2 10^{-6}(332\eta^2 - 579\eta^3 + 192\eta^5 + 252\eta^6 - 162\eta^7 - 30\eta^8 - 15\eta^9 + 10\eta^{11})/9{\cdot}072,$$

$$G_1 = (-\eta + 5\eta^4 - 4\eta^5)/20 + \alpha 10^{-4}(-8\eta - 35\eta^4 + 63\eta^5 - 20\eta^7)/1{\cdot}26,$$

$$A_1 = -54/35 + \alpha/3150 - 1{\cdot}93 \times 10^{-3}\alpha^2/5{\cdot}04 \tag{37}$$

First, a reference solution of the similarity equations (32) and (33) has been calculated. This solution is based on an AFE solution using the first order asymptotic approximation and employing a uniform mesh of 160 elements. The value of A is calculated in such a way that the boundary condition of F at $\eta = 1$ is accurately satisfied. Then, this reference value of A has been taken as a known value and the accuracy of the various schemes has been tested by comparing the error in $F^h(1)$ (note that $F(1)$ has not imposed as a boundary condition). The results are shown in Table VI. For the uncoupled case, $\alpha = 0$, and for the coupled case with $\alpha = 5$ and $Re = 1$, the AFE scheme based on the zeroth order asymptotic solution does not improve the accuracy of $F^h(1)$. This illustrates the possibility that a low order analytical approximation does not have sufficient structure to correct the truncation error.[9] For the other cases, the improvement is proportional to Re^{n+1}. Next, for each scheme, A was calculated such that $F^h(1) = 1$. The computed errors of A are

Table VI. Example 2—absolute computed errors of F^h at the rotating disk, $|F^h(1) - 1|$

		$\alpha = 0$			$\alpha = 5$		
NE	Scheme	$Re = 10^{-2}$	$RE = 10^{-1}$	$RE = 1$	$Re = 10^{-2}$	$Re = 10^{-1}$	$Re = 1$
	a	2·38E-4	2·30E-3	1·61E-2	1·88E-2	1·61E-2	3·46E-3
2	b				2.61E-4	2·53E-3	
	c	3·54E-7	3·43E-5	2·58E-3	2·99E-7	2·91E-5	1·00E-3
	a	4·63E-5	4·46E-4	3·10E-3	3·74E-3	3·21E-3	6·81E-4
4	b				5.08E-5	4·93E-4	
	c	7·54E-8	7·30E-6	5·39E-4	6·46E-8	6·28E-6	4·84E-4
	a	9·71E-6	9·36E-5	6·57E-4	7·95E-4	6·86E-4	1·43E-4
8	b				1·05E-5	1·02E-4	
	c	1·44E-8	1·40E-6	1·01E-4	1·18E-8	1·14E-6	8·79E-5
	a	2·19E-6	2·11E-5	1·50E-4	1·80E-4	1·56E-4	3·26E-5
16	b				2.33E-6	2·26E-5	
	c	2·86E-9	2·77E-7	1·98E-5	2·18E-9	2·13E-7	1·63E-5

Notation:
(a) SFE scheme.
(b) AFE scheme based on zeroth order asymptotic approximation.
(c) AFE scheme based on first order asymptotic approximation.

Table VII Example 2—computed errors of the constant of integration, A

NE	Scheme	$\alpha = 0$ Re $= 10^{-2}$	$\alpha = 0$ Re $= 10^{-1}$	$\alpha = 0$ Re $= 1$	$\alpha = 5$ Re $= 10^{-2}$	$\alpha = 5$ Re $= 10^{-1}$	$\alpha = 5$ Re $= 1$
	a	2·86E-3	2·83E-3	2·52E-1	2·26E-1	1·99E-1	5·44E-2
2	b	2·86E-3	2·83E-3	2·52E-1	3·15E-3	3·12E-2	2·91E-1
	c	4·26E-6	4·24E-4	4·07E-2	3·60E-6	3·60E-4	3·56E-2
	a	5·57E-4	5·49E-3	4·76E-2	4·50E-2	3·96E-2	1·043E-2
4	b	5·57E-4	5·49E-3	4·76E-2	6·11E-4	6·07E-3	5·67E-2
	c	9·07E-7	8·99E-5	3·02E-4	7·77E-7	7·74E-5	7·42E-3
	a	1·17E-4	1·15E-3	1·00E-2	9·59E-3	8·44E-3	2·18E-3
8	b	1·17E-4	1·15E-3	1·00E-2	1·26E-4	1·25E-3	1·19E-2
	c	1·74E-7	1·72E-5	1·55E-3	1·41E-7	1·41E-5	1·34E-3
	a	2·63E-5	2·60E-4	− 2·29E-3	2·16E-3	1·92E-3	− 4·97E-4
16	b	2·63E-5	2·60E-4	− 2·29E-3	2·79E-5	2·78E-4	2·69E-3
	c	3·44E-8	3.41E-6	3·02E-4	2·63E-7	2·62E-6	2.49E-4
e^{A^ε}	0	1·54E-2	1·54E-1	1·54	1·55E-2	1·55E-1	1·59
	1	6·40E-6	6·39E-4	6·27E-2	3·62E-6	3·65E-4	3·90E-2

Notation:
(a) SFE scheme.
(b) AFE scheme based on zeroth order asymptotic approximation.
(c) AFE scheme based on first order asymptotic approximation.

shown in Table VII. It seems that, by using the AFE scheme with a crude mesh of elements, one can get a better estimate for the value of A than both SFE and asymptotic approximations. Similar conclusions hold for F and F' (numerical results for the case of rotating compressible flows are presented in Reference 18). The pressure is proportional to A, F and F'. Hence, by applying the AFE scheme, one can significantly improve the accuracy of the pressure and velocity components.

3.3 Multi-dimensional diffusion–convection equations

Consider the following class of stiff problems:

$$Lu = -\varepsilon\nabla^2 q + L_0 q = f, \quad \text{in } \Omega, \tag{38}$$

$$q = g, \quad \text{on } \Gamma. \tag{39}$$

Here, Ω is a bounded open set in R^N with boundary Γ, L is a second order elliptic operator, and L_0 is the outer operator of reduced order. It is assumed that L_0 is a well behaved operator and f and g are assumed to be sufficiently smooth functions of $\mathbf{x}$. The 1-D version of this model problem has been presented in Section 3.1. The notation defined in that section has been adopted here and properly extended.

The weak form, Petrov–Galerkin finite element formulations of (38) are given by

$$\begin{aligned}&[\varepsilon(\nabla^{\mathrm{T}} M_j, \nabla N_i)_h + (M_j, L_0 N_i)_h] q_i^h \\ &\quad = (M_j, f)_h + \lambda\{[\varepsilon(\nabla^{\mathrm{T}} M_j, \nabla N_i)_h + (M_j, L_0 N_i)_h] q_i^\varepsilon - (M_j, f^\varepsilon)_h\},\end{aligned} \tag{40}$$

where $\lambda = 0$ for the SFE scheme and $\lambda = 1$ for the AFE scheme.

Test case—2-D diffusion–convection equations. We consider the advection of a quantity q in a square region with unit area (Figure 2).

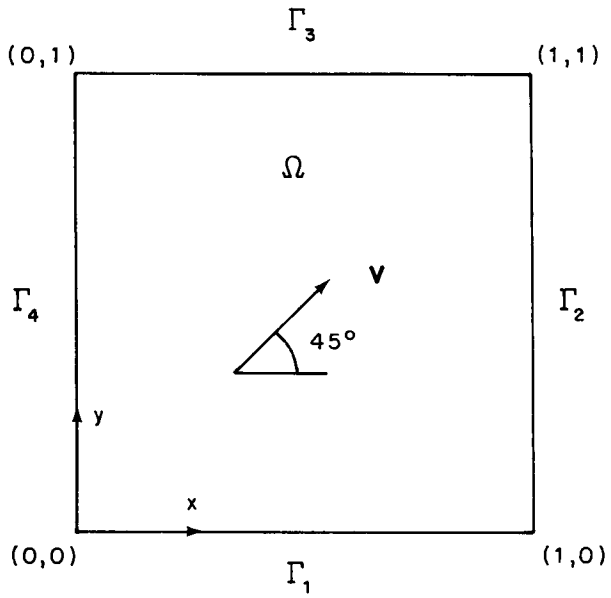

Figure 2. 2-D diffusion–convection problem—notation

The normalized equation is

$$Lq = -\varepsilon\nabla^2 q + \mathbf{V}\cdot\nabla q = 0, \quad \text{in } \Omega, \tag{41}$$

$$\begin{aligned} q &= \sin \pi x, && \text{on } \Gamma_1, \\ q &= -\sin \pi y, && \text{on } \Gamma_4, \\ q &= y + \sin \pi(1-y), && \text{on } \Gamma_2, \\ q &= x + \sin \pi(x-1), && \text{on } \Gamma_3. \end{aligned} \tag{42}$$

In the present case, the velocity field $\mathbf{V}$ is assumed to be known, and its x and y components denoted by u and v, respectively, satisfy $u = v = \sqrt{2}/2$. A similar problem is often used as a test case[6,20–22] for various numerical schemes and it is well known that most methods fail as the element Reynolds number, $Re^h \equiv |\mathbf{V}|h/\varepsilon$, becomes larger than $O(1)$.

The asymptotic solution is obtained by the method of matched asymptotic expansions.[3,4] The zeroth order asymptotic solution is given by

$$\begin{aligned} q^\varepsilon &= q_O^\varepsilon + q_I^\varepsilon \\ &= \sin \pi(x-y) + [xE_y + yE_x - E_xE_y] \end{aligned} \tag{43}$$

where

$$\begin{aligned} E_x &= (e^{ux/\varepsilon} - 1)/(e^{u/\varepsilon} - 1), \\ E_y &= (e^{vy/\varepsilon} - 1)/(e^{v/\varepsilon} - 1). \end{aligned} \tag{44}$$

Here, the outer solution, q_O^ε, remains constant along stream lines and the inner solution, q_I^ε, includes horizontal, vertical and corner boundary layers. The performance of the SFE and AFE schemes for the above example is shown in Table VIII. As we do not have the exact solution for this problem, we used as reference a solution obtained with the AFE scheme employing a uniform mesh

Table VIII. Example 3—comparison between SFE and AFE schemes

		3 × 3 el.*			6 × 6 el.			12 × 12 el.		
ε	e_∞^ε	e_∞^{SFE}	e_∞^{AFE}	K_∞	e_∞^{SFE}	e_∞^{AFE}	K_∞	e_∞^{SFE}	e_∞^{AFE}	K_∞
1/10	4·67E-1	3·10E-2	2·31E-2	1·60	4·40E-3	3·65E-3	1·78	4·51E-4	3·69E-4	1·75
1/20	3·64E-1	1·40E-1	7·06E-2	1·39	2·65E-2	1·29E-2	1·33	2·90E-3	1·81E-3	1·72
1/50	2·16E-1	5·08E-1	1·36E-1	1·24	2·24E-1	3·95E-2	0·819	4·79E-2	1·05E-2	1.01
1/100	1·27E-1	8·46E-1	1·22E-1	1·13	5·43E-1	5·91E-2	0·855	2·32E-1	2·37E-2	0·802
1/200	8·05E-2	1·41	9·88E-2	0·869	8·21E-1	5·64E-2	0·853	5·69E-1	3·28E-2	0·717
1/500	4·05E-2	3·26	8·66E-2	0·657	1·19	3·91E-2	0·810	9·18E-1	3·06E-2	0·823
1/1000	2·22E-2	6·43	8·50E-2	0·597	1·90	3·48E-2	0·826	1·07	2·22E-2	0·936

* Uniform mesh of biquadratic Lagrangian elements.

of 24 × 24 biquadratic Lagrangian elements. For this case the ratio K_∞ is defined by

$$K_\infty \equiv \frac{e_\infty^{AFE} u(1,1)}{e_\infty^{SFE} e_\infty^{\varepsilon}}. \tag{45}$$

We expect K_∞ to approach a constant for small values of h and ε. The fact that the values of K_∞ presented in the Table are not far from unity, while the values of the errors e_∞^{AFE}, e_∞^{SFE} and e_∞^{ε} vary by orders of magnitude, supports our theory also for the multi-dimensional case. We do not expect better agreement because of two reasons—the element size is not sufficiently small for the problem under consideration and the estimated errors may be affected by errors in the reference solution. The magnitude of K_i can serve as a measure of the efficiency of the AFE method. We can conclude that the AFE method is quite attractive for the present problem, since the computed values of K_∞ are not far from unity.

We have not considered graded grids because our main object is to compare the relative performance of different finite element formulations as a function of h and ε. However, it should be pointed out that improved results as compared to those quoted in the above Tables, could have been obtained by using either graded grids or an SFE scheme based on variable upwinding with adaptive mesh refinement strategies.[6,23,24]

4. CONCLUDING REMARKS

1. The theory of the AFE scheme was presented and verified in several test cases. The dependence of the error estimate on the mesh size and the small characteristic parameter is best presented via the 'constant' K_i which turns out to be close to one. It follows that the AFE method gives an error which is approximately equal to the product of errors of SFE and asymptotic approximations. Thus, the AFE method can be very useful for difficult problems of regular and singular perturbation types.

2. When an analytical asymptotic solution is available the extra work required to compute the right hand side correction is usually negligible as compared to the solution of the resulting set of algebraic equations. On the other hand solving the problem on a finer grid by the SFE method can be very costly in CPU time and memory requirements. The process of finding an analytical asymptotic solution can be performed by means of symbolic computer languages.[25] In some problems analytical solutions are difficult or impossible to achieve. In such cases numerical methods can be applied to the reduced equations,[26,27] which are usually easier to solve than the

original system (the reduced equations may be of lower order or lower dimension than the original set).

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